Please feel free to e-mail any corrections, thoughts, alternative solutions, etc. to p.freyne@btinternet.com

MAY/JUNE 1972 A.E.B. A-LEVEL PURE MATHS S PAPER

IF ANYBODY LIKES DISCUSSING MATHS PROBLEMS LIKE THESE, HAS ANY CORRECTIONS TO THESE ANSWERS, OR SOLUTIONS TO 3(i) & 7(b) SPECIFICALLY PLEASE E-MAIL ME AT pfreyne@btinternet.com

1(a) (i) $dx/dt = 2 + 2\cos(2t)$ $\quad dy/dt = -2\sin(2t)$

$dy/dx = dy/dt \times dt/dx = (dy/dt) \times (dx/dt)^{-1} = -2\sin(2t)/2(1+\cos 2t)$

$= -2\sin(t)\cos(t)/(1+\cos^2(t) - \sin^2(t)) = -2\sin(t)\cos(t)/2\cos^2(t)$

$= -\tan(t)$ Q.E.D.

(ii) Say $w = dy/dx$ and $u = dx/dy$

$$\frac{d^2y}{dx^2} = \frac{dw}{dx} = \frac{dw}{dt} \times \frac{dt}{dx} = \frac{d}{dt}(-\tan(t)) \cdot \frac{1}{2(1+\cos(2t))}$$

$= -\sec^2(t)/2(1+\cos 2t)$

$$\frac{d^2x}{dy^2} = \frac{du}{dy} = \frac{d}{dt}\left(\frac{-1}{\tan(t)}\right)\frac{1}{-2\sin(2t)}$$

$$= \frac{1}{\sin^2(t)} \cdot \frac{1}{(-2\sin(2t))}$$

$$\therefore (d^2y/dx^2)/(d^2x/dy^2) = -\frac{1}{\cos^2(t)} \cdot \frac{1}{\cancel{2.2.}\cos^2(t)} \cdot (-\sin^2 t . \cancel{2.2.}\cos t \sin t)$$

$$= \frac{+\sin^3 t \cos t}{\cos^4(t)} = +\tan^3(t)$$ Q.E.D.

1(b). At P, $\frac{dy}{dx} = a_1 + 2a_2 x_1 + 3a_3 x_1^2$

At Q, $\frac{dy}{dx} = a_1 + 2a_2 x_2 + 3a_3 x_2^2$

and since $\left(\frac{dy}{dx}\right)_P = \left(\frac{dy}{dx}\right)_Q$

$$a_1 + 2a_2 x_1 + 3a_3 x_1^2 = a_1 + 2a_2 x_2 + 3a_3 x_2^2$$

or $3a_3 (x_2^2 - x_1^2) = 2a_2 (x_1 - x_2)$.

giving $3a_3 (x_2 - x_1).(x_2 + x_1) = -2a_2 (x_2 - x_1)$

and eventually $x_2 + x_1 = -2a_2/3a_3$ Q.E.D

Given that R.H.S. of the equation for the curve takes the form of a cubic polynomial, I would suggest that it has a symmetric sigmoid shape and so the abscissa for the point of inflexion would come at $(x_1 + x_2)/2 = -a_2/3a_3$!?

My thoughts AN INTERESTING & CHALLENGING QUESTION, I FELT, EVEN IF IT FELT A LITTLE LIKE "STANDARD BOOKWORK," IN PLACES.

2(a) Try for $(n+1)$, that is, $\frac{d^{n+2}}{dx^{n+2}}\left(x^{n+1}e^{1/x}\right)$

which is the same as $\frac{d}{dx}\left(\frac{d^{n+1}}{dx^{n+1}}\left(x.(x^n e^{1/x})\right)\right)$

which equals $\frac{d}{dx}\left(x.\frac{d^{n+1}}{dx^{n+1}}(x^n e^{1/x}) + (n+1)\frac{d^n}{dx^n}(x^n e^{1/x})\right)$

(Arrived at using Leibnitz's Theorem* with $u = x^n e^{1/x}$ and $v = x$)

$$= x\frac{d^{n+2}}{dx^{n+2}}(x^n e^{1/x}) + \frac{d^{n+1}}{dx^{n+1}}(x^n e^{1/x}) + (n+1)\frac{d^{n+1}}{dx^{n+1}}(x^n e^{1/x})$$

(First two terms arrived at using Product Rule)

$$= x\frac{d}{dx}\left(\frac{d^{n+1}}{dx^{n+1}}\left(x^n e^{1/x}\right)\right) + (n+2)\frac{d^{n+1}}{dx^{n+1}}\left(x^n e^{1/x}\right)$$

$$= x\frac{d}{dx}\left((-1)^{n+1}\frac{e^{1/x}}{x^{n+2}}\right) + (n+2)\left((-1)^{n+1}.\frac{e^{1/x}}{x^{n+2}}\right)$$

FROM INITIAL DEFINITION IN QUESTION.

$$= (-1)^{n+1}x\left(\frac{1}{x^{n+2}}.e^{1/x}.\left(-\frac{1}{x^2}\right) - (n+2)\frac{1}{x^{n+3}}e^{1/x}\right) + (n+2)(-1)^{n+1}\frac{e^{1/x}}{x^{n+2}}$$

which after "tidying up" gives:- $(-1)^{((n+1)+1)}.\frac{1}{x^{((n+1)+2)}}.e^{1/x}$

True for $n=1$? i.e. does $\frac{d^2(xe^{1/x})}{dx^2} = \frac{e^{1/x}}{x^3}$?

First derivative $= \left(-\frac{1}{x}\right)e^{1/x} + e^{1/x}$

Q.E.D.

Second derivative $= e^{1/x}.\left(\frac{1}{x^3}\right) + \cancel{e^{1/x}\left(\frac{1}{x^2}\right)} - \cancel{e^{1/x}\left(\frac{1}{x^2}\right)}$

My Thoughts

A REALLY CHALLENGING QUESTION OF SOME ELEGANCE
I VERY MUCH ENJOYED FINDING A SOLUTION TO THIS.
PROVIDES AN OPPORTUNITY ~~for~~ FOR USE OF *Leibnitz's Theorem
III (WHICH REMAINS, I PRESUME, ONE OF THE MORE ADVANCED A-LEVEL TOPICS)

2(b) Using the *formula for addition of arc tangents.

$$\tan^{-1}(x) + \tan^{-1}(c/x) = \tan^{-1}\left(\frac{(x + c/x)}{\left(1 - x.\frac{c}{x}\right)}\right) = \frac{3\pi}{4}.$$

Which gives $(x + c/x)/(1-c) = -1$ (by taking the tangent of both sides)

or $x + c/x + (1-c) = 0$ leading to $x = \frac{-(1-c) \pm \sqrt{(1-c)^2 - 4c}}{2}$

If those two roots differ by 2 then we have.

$$\left(\frac{-(1+c) + \sqrt{(1-c)^2 - 4c}}{2}\right) - \left(\frac{-(1+c) - \sqrt{(1-c)^2 - 4c}}{2}\right) = 2.$$

that is, $\sqrt{(1-c)^2 - 4c} = 2.$

and eventually $c^2 - 6c - 3 = 0$ $c = 3 \pm 2\sqrt{3}$

and substituting back into the formulae for the two roots for the equation in x gives

$x = \sqrt{3}$ or $2 + \sqrt{3}$. (for $c = 3 + 2\sqrt{3}$)

(BUT LEADS TO NEGATIVE RESULT. or $x = -\sqrt{3}$ or $2 - \sqrt{3}$. for $c = 3 - 2\sqrt{3}$)

BY WAY OF A CHECK Putting in original formula: $\tan^{-1}(\sqrt{3}) + \tan^{-1}\left(\frac{3 + 2\sqrt{3}}{\sqrt{3}}\right) = \frac{3\pi}{4}$ radians

$\tan^{-1}(2 + \sqrt{3}) + \tan^{-1}\left(\frac{3 + 2\sqrt{3}}{2 + \sqrt{3}}\right) = \frac{3\pi}{4}$ radians

Q.E.D.

My Thoughts: AN INTERESTING AND ENJOYABLE QUESTION, REQUIRING EITHER MEMORISATION OR DERIVATION (UNDER EXAM CONDITIONS!!) OF THE *FORMULA FOR ADDITION OF ARC TANGENTS AND ARITHMETIC MANIPULATION OF SURDS (WHILE KEEPING A CLEAR HEAD!)

3(i) * NOT SURE ABOUT THIS ONE! UNLIKE PART (ii), IT <u>DIDN'T</u> SEEM TO ME TO BE A CASE OF RECOGNISING KNOWN "STANDARD" SERIES (EVEN IF ** $\log_e 2$ <u>SEEMS</u> TO FEATURE IN THE ANSWER).

I TRIED LOOKING AT AS: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)}$

AND RESOLVING $\dfrac{1}{(4k+1)(4k+2)(4k+3)} \left(= \dfrac{1}{2}\dfrac{1}{(4k+1)} - \dfrac{1}{(4k+2)} + \dfrac{1}{2}\dfrac{1}{(4k+3)} \right)$

INTO PARTIAL FRACTIONS! <u>HOWEVER</u>, YOU GET

$$
\begin{array}{llll}
(k=0) & \frac{1}{2} & -\frac{1}{2} & +\frac{1}{6} \\
(k=1) & \frac{1}{10} & -\frac{1}{6} & +\frac{1}{14} \\
(k=2) & \frac{1}{18} & -\frac{1}{10} & +\frac{1}{22} \\
(k=3) & \frac{1}{26} & -\frac{1}{14} & +\frac{1}{30} \\
(k=4) & \frac{1}{34} & -\frac{1}{18} & \vdots \\
 & \vdots & \vdots & \ldots
\end{array}
$$

WHERE THE NEGATIVE SECOND TERM "KNOCKS OUT" ALTERNATE THIRD AND FIRST TERMS TENDING, IN THE LONG RUN, TO AN OVERALL <u>ZERO SUM</u>! ON THE OTHER HAND <u>MAPLE</u> (I HAVE TO ADMIT) GIVES AN ANSWER OF ** $(1/4)\log_e 2$ AND OTHER EXPERIMENTS THAT I'VE DONE USING SPREADSHEETS/APPROXIMATIONS SEEM TO CONFIRM THAT AS THE RIGHT ANSWER!
ANY THOUGHTS, ANYONE?

3(ii) Multiplying the terms of the series out gives:

$$x + \frac{x^2}{1!} + \frac{2x}{1!} + \frac{x^2}{2!} + \frac{4x^2}{2!} + \frac{x^4}{3!} + \frac{6x^3}{3!} + \cdots$$

Now group terms to give:

$$\left(x + \frac{x^2}{1!} + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots\right) + \left(\frac{2x}{1!} + \frac{4x^2}{2!} + \frac{6x^3}{3!} + \cdots\right)$$

$$= x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + 2x\left(1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots\right)$$

$$= xe^x + 2xe^x = 3xe^x$$

My Thoughts

I FOUND PART (i) TO BE QUITE PERPLEXING!! WHILE PART (ii) IS A GOOD QUESTION IT TENDS TO ERR ON THE SIDE OF "STANDARD BOOKWORK".

4(a) Say $x = a_1 y + a_2 y^2 + a_3 y^3 + \dots$ (as suggested by the "successive approximation")

$$\frac{dx}{dy} = a_1 + 2a_2 y + 3a_3 y^2 + \dots$$

but $\left(\frac{dx}{dy}\right) = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{(1 - x + x^2/2! - x^3/3!)}$

and for $x=0$, $y=0$ giving $a_1 = 1$.

So for small enough x $x = a_1 y = y$

Similarly $\frac{d^2x}{dy^2} = 2a_2 + 6a_3 y$

and if we let $\frac{dx}{dy} = z$ $\frac{d^2x}{dy^2} = \frac{dz}{dy} = \frac{dz}{dx} \cdot \frac{dx}{dy}$

$$= \frac{-1}{(1 - x + x^2/2! - x^3/3!)^2} \cdot (-1 + x - x^2/2!) \cdot \frac{1}{(1 - x + x^2/2! - x^3/3!)}$$

again for $x=0$, $y=0$ giving $2a_2 = 1$ $a_2 = 1/2$

and our next approximation for small x, $x = y + y^2/2$.

Finally, $\frac{d^3x}{dy^3} = 3.2.a_3$, $\frac{d^2x}{dy^2} = v$, $\frac{dv}{dy} = \frac{dv}{dx} \cdot \frac{dx}{dy}$.

$$= \frac{d}{dx}\left(\frac{-(-1 + x - x^2/2!)}{(1 - x + x^2/2! - x^3/3!)^3}\right) \cdot \frac{1}{(1 - x + x^2/2! - x^3/3!)}$$

Leading eventually to $a_3 = 1/3$ (on putting $x=y=0$)

with the final approximation $x = y + y^2/2 + y^3/3$

Q.E.D.

4(b) $\frac{(x+1)}{(x-n+1)} = 1 + \frac{n}{x} \cdot \frac{x}{(x-n+1)}$ (given!)

$$= 1 + \frac{n}{x}\left(\frac{(x+1-1)}{(x-n+1)}\right) = 1 + \frac{n}{x}\left(\frac{(x+1)}{(x-n+1)} - \frac{1}{(x-n+1)}\right)$$

$$= 1 + \frac{n}{x}\left(1 + \frac{n}{x}\frac{1}{(x-n+1)} - \frac{1}{x-(n+1)}\right)$$

$$= 1 + \frac{n}{x} + \frac{n}{x}\left(\frac{(n-1)}{(x-n+1)}\right)$$

$$= 1 + \frac{n}{x} + \frac{n(n-1)}{x}\left(\frac{(x-1).1}{(x-1).(x+1-n)}\right)$$

$$= 1 + \frac{n}{x} + \frac{n(n-1)}{x(x-1)}\left(\frac{(x+1)-2}{(x+1-n)}\right)$$

$$= 1 + \frac{n}{x} + \frac{n(n-1)}{x(x-1)}\left(1 + \frac{n}{x}\frac{x}{(x-n+1)} - \frac{2}{(x-n+1)}\right).$$

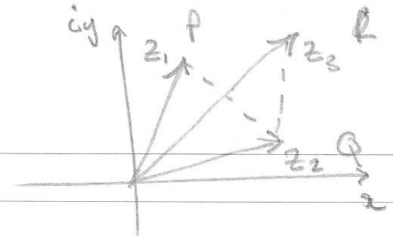
$$= 1 + \frac{n}{x} + \frac{n(n-1)}{x(x-1)} + \frac{n(n-1)}{x(x-1)}(n-2).\frac{(x-2)}{(x-2)}.\frac{1}{(x-n+1)}$$

etc.

Q.E.D.

My thoughts: A VERY CHALLENGING QUESTION. HOPE I DIDN'T PLAY TOO "FAST & LOOSE" WITH TRUNCATION OF POWER SERIES FOR PART (a). PART (b) REALLY TRICKY! A VERY ELEGANT ASPECT OF IT WAS, I FEEL, A NEED TO "BUILD IN REDUNDANCY" IN THE NUMERATOR OF SOME FRACTIONS BY EXPRESSING x as $(x+1-1)$

VIII



IN VECTOR TERMS

5(a) $\underrightarrow{QP} = -(z_2 - z_1)$, $\underrightarrow{QR} = z_3 - z_2$.

For PQR to be a right angle, the scalar (dot) product of $\underrightarrow{QP}$ and $\underrightarrow{QR}$ needs to be zero.

But for two complex nos. ω_1 and ω_2 the scalar product can be defined by
$\omega_1 \cdot \omega_2 = Re(\bar{\omega}_1 \omega_2)$.

So $(z_3 - z_2)\cdot(z_2 - z_1) = Re\left(\overline{(z_3 - z_2)}(z_2 - z_1)\right)$

$= Re\left(\frac{(|z_3 - z_2|)^2}{(z_3 - z_2)}(z_2 - z_1)\right)$ and since $|z_3 - z_2|^2$ is by definition real, if $Re\left((|z_3 - z_2|)^2 \frac{(z_2 - z_1)}{(z_3 - z_2)}\right) = 0$

$Re\left(\frac{(z_2 - z_1)}{(z_3 - z_2)}\right) = \phi$ or $\frac{z_2 - z_1}{z_3 - z_2}$ is purely imaginary Q.E.D.

(b) $\cosh x = (e^{x} + e^{-x})/2 = (e^{-i(ix)} + e^{i(ix)})/2$.

$= (\cos(ix) - i\sin ix + \cos ix + i\sin(ix))/2 = \cos(ix)$ Q.E.D.

$i\sinh x = i(e^{x} - e^{-x})/2 = i(e^{-i(ix)} - e^{i(ix)})/2$

$= i(\cos(ix) - i\sin(ix) - \cos ix - i\sin ix)/2 = (-i)^2\sin(ix)$

$= \sin(ix)$ Q.E.D.

$$\operatorname{Tan}(x+iy) = \frac{\operatorname{Sin}(x+iy)}{\operatorname{Cos}(x+iy)} = \frac{\operatorname{Sin}(x)\operatorname{Cos}(iy)+\operatorname{Cos}(x)\operatorname{Sin}(iy)}{\operatorname{Cos}x\operatorname{Cos}(iy)-\operatorname{Sin}x\operatorname{Sin}(iy)}$$

$$= \frac{\operatorname{Sin}x\operatorname{Cosh}y + i\operatorname{Cos}x\operatorname{Sinh}y}{\operatorname{Cos}x\operatorname{Cosh}y - i\operatorname{Sin}x\operatorname{Sinh}y}$$

multiplying by the complex conjugate of the denominator gives:

$$\frac{(\operatorname{Sin}x\operatorname{Cosh}y + i\operatorname{Cos}x\operatorname{Sinh}y).(\operatorname{Cos}x\operatorname{Cosh}y + i\operatorname{Sin}x\operatorname{Sinh}y)}{(\operatorname{Cos}^2x\operatorname{Cosh}^2y - \operatorname{Sin}^2x\operatorname{Sinh}^2y)}$$

$$= \frac{\operatorname{Sin}x\operatorname{Cos}x\operatorname{Cosh}^2y - \operatorname{Cos}x\operatorname{Sin}x\operatorname{Sinh}^2y}{(\operatorname{Cos}^2x\operatorname{Cosh}^2y - \operatorname{Sin}^2x\operatorname{Sinh}^2y)} + i(\ldots)$$

The real part of which is:-

$$\frac{\operatorname{Sin}x\operatorname{Cos}x(\operatorname{Cosh}^2y - \operatorname{Sinh}^2y)}{(\operatorname{Cos}^2x\operatorname{Cosh}^2y + (\operatorname{Cosh}^2y-1)\operatorname{Sin}^2x)} = \frac{\operatorname{Sin}x\operatorname{Cos}x}{\left(\left(\frac{\operatorname{Cosh}(2y)+1}{2}\right)+\left(\frac{\operatorname{Cos}(2x)-1}{2}\right)\right)}$$

$$= \frac{\operatorname{Sin}(2x)}{(\operatorname{Cosh}(2y) + \operatorname{Cos}(2x))}$$

Q.E.D

My Thoughts

ABOVE & BEYOND HOPING THAT MY ANSWER TO PART (a) "STANDS UP", A GOOD QUESTION WHICH MAY REQUIRE "STANDARD BOOKWORK" BUT WHICH DRAWS ON MORE ADVANCED TOPICS IN THE A-LEVEL SYLLABUS.

6(a) Letting $u = \sin x \quad du = \cos x\,dx$

$$I = \int_0^{\pi/2} \frac{\cos x\,dx}{(6 + \sin x - \sin^2 x)} = \int_0^1 \frac{du}{(6 + u - u^2)} = -\int_0^1 \frac{du}{(u^2 - u - 6)}$$

Resolving $\frac{1}{(u^2-u-6)}$ into partial fractions gives

$$-\frac{1}{5(u+2)} + \frac{1}{5(u-3)}, \quad \text{so } I = \frac{1}{5}\left(-\int_0^1 \frac{du}{(u-3)} + \int_0^1 \frac{du}{(u+2)}\right)$$

$$= \frac{1}{5}\left(-\log_e(2/3) + \log_e(3/2)\right) = \frac{1}{5}\log_e\left(\frac{3}{2}\right)^2 = \frac{2}{5}\log_e\left(\frac{3}{2}\right)$$

Q.E.D.

6(b) Let $z = 1/x \quad dx = -dz/z^2$

$$I = \int_0^\infty \frac{dx}{(1+x^4)} = \int_\infty^0 -\frac{dz}{z^2}\cdot\frac{1}{(1 + 1/z^4)} = \int_0^\infty \frac{z^2\,dz}{(1+z^4)}$$

$$= \int_0^\infty \frac{x^2\,dx}{(1+x^4)} \quad \text{Q.E.D.}$$

$$(1 + \sqrt{2}x + x^2).(1 - \sqrt{2}x + x^2) = (1 + x^4)$$

$$\text{So } \int_0^\infty \frac{(1 + \sqrt{2}x + x^2)\,dx}{(1+x^4)} = \int_0^\infty \frac{dx}{(1 - \sqrt{2}x + x^2)} = I_1, \text{ say}$$

$$\text{and } \int_0^\infty \frac{(1 - \sqrt{2}x + x^2)\,dx}{(1+x^4)} = \int_0^\infty \frac{dx}{(1 + \sqrt{2}x + x^2)} = I_2, \text{ say}$$

But $I_1 + I_2 = 2\left(\int_0^\infty \frac{dx}{(1+x^4)} + \int_0^\infty \frac{x^2\,dx}{(1+x^4)}\right) = 4\int_0^\infty \frac{dx}{(1+x^4)}$

$$= \int_0^\infty \frac{dx}{(1-\sqrt{2}x+x^2)} + \int_0^\infty \frac{dx}{(1+\sqrt{2}x+x^2}$$

$$= \int_0^\infty \frac{dx}{\left(\left(x-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2\right)} + \int_0^\infty \frac{dx}{\left(\left(x+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2\right)}$$

$$= \frac{1}{\sqrt{2}}\left[\mathrm{Tan}^{-1}\left(\frac{x-1/\sqrt{2}}{1/\sqrt{2}}\right) + \mathrm{Tan}^{-1}\left(\frac{x+1/\sqrt{2}}{1/\sqrt{2}}\right)\right]_0^\infty$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - \mathrm{Tan}^{-1}(-1) + \frac{\pi}{2} - \mathrm{Tan}^{-1}(1)\right) = \frac{\pi}{\sqrt{2}\cdot 2} \quad \text{Q.E.D.}$$

7(a) $I_{n+1} + I_n = \int_0^{\pi/2} \frac{\cos((2n+1)x)}{\cos x}\,dx + \int_0^{\pi/2} \frac{\cos((2n-1)x)}{\cos x}\,dx$

$$= \int_0^{\pi/2} \left(\cos(2nx)\cos x - \sin(2nx)\sin x + (\cos(2nx)\cos x + \sin(2nx)\sin x)\right)dx$$

$$= 2\int_0^{\pi/2} \cos(2nx)\,dx = \left[\frac{1}{n}\sin(2nx)\right]_0^{\pi/2} = \emptyset$$

$\therefore \quad I_{n+1} = -I_n \;,\; I_n = -I_{(n-1)} \;,\; I_{(n-1)} = -I_{(n-2)}$

Giving $\quad I_n = (-1)^k I_{(n-k)}$

and $\quad I_n = (-1)^{n-1} I_{(n-(n-1))} = (-1)^{n-1} I_1 = (-1)^{n-1}\int_0^{\pi/2} \frac{\cos x}{\cos x}\,dx$

$$= (-1)^{(n-1)}\frac{\pi}{2} \quad \text{Q.E.D.}$$

XII

7(b) I'M NOT SURE I UNDERSTAND THIS PART QUESTION, ABOVE AND BEYOND THE PROBABLY NAIVE PERCEPTION THAT IT LOOKS LIKE AN "OBVIOUS" RESULT! I'M GUESSING THE SOLUTION LEANS HEAVILY TOWARDS THE PURE & ANALYTICALLY RIGOROUS!

IF ANYBODY CARES TO PROVIDE A SOLUTION I WOULD BE MOST INTERESTED! PLEASE LET ME KNOW, IF SO!

8(a) $$\frac{d}{dx}\left(\frac{x^2+3x+3}{(x+1)^2}\right) = (x^2+3x+3)\frac{d}{dx}\left((x+1)^{-2}\right) + \frac{1}{(x+1)^2}\frac{d}{dx}\left(x^2+3x+3\right)$$

$$= (x^2+3x+3).\frac{-2}{(x+1)^3} + \frac{1}{(x+1)^2}.(2x+3)$$

for a turning point this derivative is zero, giving

$2x^2+6x+6 = 2x^2+5x+3$. (after some cross-multiplication)

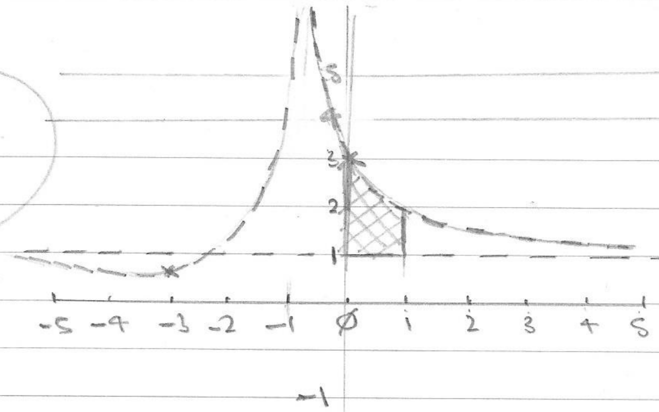
or $x=-3$, which is presumably the minimum referred to in the question, although finding the sign of the second derivative could act as a rigorous check!

In terms of sketching the curve; function has:
A minimum at -3, at which point the function $= 0.75$
A value of 3 for $x = 0$
A tendency to "blow up" around $x=-1$.
And the asymptotes may be found by dividing both its numerator and denominator by x^2

i.e. $\frac{1/x^2(x^2+3x+3)}{1/x^2(x+1)^2} = \frac{1+3/x+3/x^2}{(1+1/x)^2}$

which is equal to 1 if x is put to either $\pm\infty$

So the graph looks something like:-
(ROUGH DRAWING PERMITTING)



Area to be found is the cross-hatched area in the rough graph, which will be:

$$\int_0^1 \frac{x^2+3x+3.}{(x+1)^2} - 1$$ ← (The square area between $x=0,\ x=1,\ y=0,\ y=1$)

$$\frac{x^2+3x+3}{(x+1)^2} = \frac{x^2+3x+3.}{x^2+2x+1}$$

Dividing out.

$$\begin{array}{r} 1 \\ x^2+2x+1 \overline{\big) \; x^2+3x+3.} \\ \underline{x^2+2x+1} \\ x+2 \end{array}$$ Remainder.

$$= 1 + \frac{x+2}{(x^2+2x+1)} = 1 + \frac{(x+1)}{(x+1)^2} + \frac{1}{(x+1)^2}$$

$$= 1 + \frac{1}{(x+1)} + \frac{1}{(x+1)^2}.$$ Integrating:

$$\int_0^1 \left(1 + \frac{1}{(x+1)} + \frac{1}{(x+1)^2}\right) dx = \left[x + \mathrm{Log}(x+1) - \frac{1}{(x+1)} \right]_0^1$$

$= \mathrm{Log}_e 2 + 3/2$ $\therefore$ Area is $\mathrm{Log}_e 2 + 3/2 - 1 = \mathrm{Log}_e 2 + 1/2$

My Thoughts

WITHOUT WISHING TO UNDERESTIMATE THIS QUESTION (PARTICULARLY SINCE MINE MIGHT NOT BE THE PERFECT ANSWER) I DID FEEL THIS WAS MORE LIKE A "STANDARD" A LEVEL QUESTION (WITH, PERHAPS, THE EXCEPTION OF THE INTEGRAL)!

XIV

9(a) $\dfrac{d}{dx}\left(\dfrac{(x+A)}{(x+B)}\right) = -(x+A) \cdot \dfrac{1}{(x+B)^2} + \dfrac{1}{(x+B)}$

$= \dfrac{(-x-A+x+B)}{(x+B)^2} = \dfrac{B-A}{(x+B)^2}$ and so

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{(B-A)}{(x+B)^2}\right) = \dfrac{-2(B-A)}{(x+B^3}$

<u>but</u> $2\left(\dfrac{dy}{dx}\right)^2 = \dfrac{2.(B-A)^2}{(x+B)^4} = \dfrac{-(B-A)}{(x+B)} \cdot \dfrac{d^2y}{dx^2}$

$= \dfrac{d^2y}{dx^2}\left(\dfrac{(x+A)-(x+B)^*}{(x+B)}\right) = \dfrac{d^2y}{dx^2}\left(\dfrac{(x+A)}{(x-B)} - 1\right) = \dfrac{d^2y}{dx^2}(y-1)$ Q.E.D

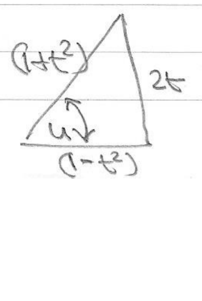
* Note again interesting requirement to "build in redundancy" (using $A-B = (x+A-x-B)$, $x-x=\emptyset$) in following this solution (as noted for 4(b).

9(b) $\dfrac{du}{dx} = 1 - \dfrac{dy}{dx}$ and so $\dfrac{dy}{dx} = \sec(x-y)$ transforms to:

$1 - \dfrac{du}{dx} = \operatorname{Sec}(u)$, $\dfrac{du}{dx} = \dfrac{(\operatorname{Cos}(u)-1)}{\operatorname{Cos}(u)}$

$\int dx = \int \dfrac{\operatorname{Cos} u \, du}{(\operatorname{Cos} u - 1)}$ an integral which is susceptible to

the use of the substitution(s) $\operatorname{Cos}(u) = (1-t^2)/(1+t^2)$, $\sin(u) = 2t/(1+t^2)$, $\operatorname{Tan}(u) = 2t/(1-t^2)$,
$t = \operatorname{Tan}\left(\frac{u}{2}\right)$ $du = \dfrac{2dt}{(1+t^2)}$



Then $\int \frac{\cos u\, du}{(\cos u - 1)} = \int \frac{(1-t^2)}{(1+t^2)} \cdot \frac{2dt}{(1+t^2)} \cdot \frac{(1+t^2)}{-2t^2} = \int \frac{-(1-t^2)\,dt}{(t^2+1)t^2}$

$= 2\int \frac{dt}{(1+t^2)} - \int \frac{dt}{t^2} = 2\,\mathrm{Tan}^{-1}(t) + \frac{1}{t} + C$

$x = 2\,\mathrm{Tan}^{-1}\left(\mathrm{Tan}\left(\frac{u}{2}\right)\right) + \mathrm{Cot}\left(\frac{u}{2}\right) + C$

$= u + \mathrm{Cot}\left(\frac{u}{2}\right) + C = x - y + \mathrm{Cot}\left(\frac{x-y}{2}\right) + C$

and eventually $y = \mathrm{Cot}\left((x-y)/2\right) + c$.

COULD/SHOULD BE SIMPLIFIED FURTHER (TO GET EXPLICIT EXPRESSION FOR Y)?

9(c) Re-arranging the equation gives:

$\frac{dy}{dx} + y\frac{(x+1)}{x} = \frac{(2+3x)}{x}$ A typical "integrating factor" equation

The integrating factor of which is $e^{\int\left(\frac{x+1}{x}\right)dx}$

$= e^{x + \log_e x} = xe^x$

Multiplying through by this integrating factor gives

$xe^x\frac{dy}{dx} + y(x+1)e^x = (2+3x)e^x$

or $\frac{d}{dx}(yxe^x) = (2+3x)e^x$

$$yxe^x = \int (2+3x)e^x\,dx = 2e^x + 3(xe^x - e^x)$$

Using integration by parts.

Giving, eventually $y = 3 - \dfrac{1}{x} + \dfrac{Ce^{-x}}{x}$ Q.E.D.

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10 WITHOUT WISHING TO BE AWKWARD, I'M NOT SURE ABOUT THE WORDING OF THIS QUESTION! FOR THE FIRST PART, I WILL ASSUME THAT "FIND THE IMAGE OF THE STRAIGHT LINE .. IN THE PLANE" AMOUNTS TO FINDING THE REFLECTION OF THE GIVEN LINE IN THE GIVEN PLANE. HERE GOES WITH MY ATTEMPT AT A SOLUTION:

WORKING ON THE BASIS THAT ANY STRAIGHT LINE CAN BE DEFINED BY TWO POINTS (THAT IT PASSES THROUGH) AND THAT THE GIVEN LINE AND ITS REFLECTION WILL COINCIDE WHERE THEY INTERSECT WITH THE PLANE, ~~THE TWO~~ POINTS I INTEND TO USE ARE THE POINT OF INTERSECTION OF THE LINES WITH THE PLANE AND THE REFLECTION OF AN ARBITRARY POINT ON THE GIVEN LINE IN THE PLANE.

EXPRESSED IN VECTOR TERMS, THE EQUATIONS FOR THE LINE AND PLANE ARE:

$r_L = -i + 2j + 4k + \lambda(3i - j + 2k)$ LINE.

and

$r \cdot (i + 2j - 4k) = 1$

By putting r_L into the equation of the plane it should be possible to find the value of λ corresponding to their point of intersection (assuming they do intersect!)

$r_L \cdot (i + 2j - 4k) = 1$ amounts to

$((3\lambda-1)i + (2-\lambda)j + (4+2\lambda)k) \cdot (i + 2j - 4k) = 1$.

$3\lambda - 1 + 4 - 2\lambda - 16 - 8\lambda = 1$, $-7\lambda = +14$, $\lambda = -2$

and putting $\lambda = -2$ into r_L gives $-7i + 4j$

which is a point in the plane since

$(-7i + 4j) \cdot (i + 2j - 4k) = 1$

Thus point of intersection is $(-7i + 4j)$

Now to find the reflection of an arbitrary point in the plane. Let's choose the point on the given line where $\lambda = 0$ i.e. $r = -i + 2j + 4k$.

To find its reflection we can drop a perpendicular from it to the plane and then extend that perpendicular by the same distance as the point is from the plane.

line representing this perpendicular will be given by $r_p = -i + 2j + 4k + \mu(i + 2j - 4k)$.

since, from the equation for the plane the vector $(i + 2j - 4k)$ is perpendicular to it!

This perpendicular will intersect the plane where
$((\mu-1)i + (2\mu+2)j + (4-4\mu)k) \cdot (i + 2j - 4k) = 1$
giving $\mu - 1 + 4\mu + 4 - 16 + 16\mu = 1$ $\quad \mu = 14/21 = 2/3$.

and $-\frac{i}{3} + \frac{10}{3}j + \frac{4}{3}k$ is a point in the plane

since $\frac{1}{3}(-i + 10j + 4k).(i + 2j - 4k) = 1$

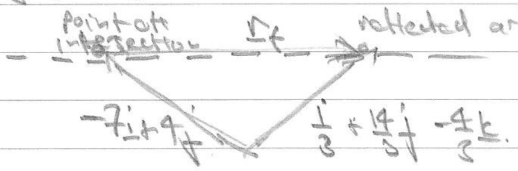
So reflection of $-i + 2j + 4k$ will be where

$\mu = \frac{4}{3}$ i.e. at $-i + 2j + 4k + \frac{4}{3}(i + 2j - 4k)$

or at $\frac{i}{3} + \frac{14}{3}j - \frac{4}{3}k$

So reflected line r_f will go through points

$\frac{i}{3} + \frac{14}{3}j - \frac{4}{3}k$ and $-7i + 4j$



~~and will take the form~~ $\cancel{r_f = \frac{1}{3}(i + 14j - 4k) + \gamma(-22i - 2j + 4k)}$

and this line intersects the plane where $\gamma = 1$
and represents the reflected point for $\gamma = 0$.

SO I'M GOING TO SAY IMAGE LINE IS:

$r_f = \frac{1}{3}(i + 14j - 4k + \gamma(-22i - 2j + 4k))$

AND HOPE I'M RIGHT!

FOR THE SECOND PART, AGAIN, I'M NOT SURE ABOUT THE WORDING "THE PLANE PASSING THROUGH THE GIVEN LINE" !?

I'M GOING TO INTERPRET IT AS ASKING FOR THE PLANE THAT "CONTAINS" THE GIVEN LINE (I.E. THE GIVEN LINE IS IN THE DESIRED PLANE) AND THE PERPENDICULAR TO THE GIVEN PLANE. IN WHICH CASE I WOULD SUGGEST:

The normal to the desired plane is given by the cross product of the unit vector in the direction of the given line and the unit vector perpendicular to the given plane, namely:

$$\frac{1}{\sqrt{14}}\cdot(3i - j + 2k) \wedge \frac{1}{\sqrt{21}}(i + 2j - 4k)$$

$$= \frac{1}{\sqrt{294}} \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 1 & 2 & -4 \end{vmatrix} = \frac{1}{\sqrt{294}}(14j + 7k)$$

$$= \sqrt{\frac{2}{3}}\left(j + \frac{k}{2}\right).$$ giving unit normal as $\frac{2}{\sqrt{5}}\left(j + \frac{k}{2}\right)$ (to desired plane)

So equation of desired plane is:

$r \cdot 2/\sqrt{5}\,(j + k/2) = p$ where r is any vector in the plane and p is the perpendicular distance from the origin to the desired plane.

$r = -i + 2j + 4k$ is a point in the plane (SINCE IT'S ON THE GIVEN LINE!)

so $(-i + 2j + 4k)\cdot\frac{2}{\sqrt{5}}\left(j + \frac{k}{2}\right) = p = 8/\sqrt{5}$

$$r \cdot \frac{2}{\sqrt{5}}\left(j + \frac{k}{2}\right) = \frac{8}{\sqrt{5}}$$

XX

$$r \cdot \left(j + \frac{k}{2}\right) = 4$$

HOPE THIS IS RIGHT AND NOT TOO MESSY !